CHAPTER-2 – COMPLEX NUMBERS

1. Find the value of the real numbers x and y, if the complex number (2 + i)x + (1 – i)y + 2i – 3 and x + (-1 + 2i)y + 1 + i are equal.
2. Find the values of the real numbers x and y, if the complex numbers (3 – i)x – (2 – i)y + 2i + 5 and 2x + (-1 + 2i)y + 3 + 2i are equal.
3. Show that $\left(2+i\sqrt{3}\right)^{10} + \left(2-i\sqrt{3}\right)^{10}$ is real and $\left(\frac{19+9i}{5-3i}\right)^{15} - \left(\frac{8+i}{1+2i}\right)^{15}$ is purely imaginary.
4. The complex numbers u, v, and w are related by $\frac{1}{u} = \frac{1}{v} + \frac{1}{w}$. If v = 3 – 4i and w = 4 + 3i, find u in rectangular form.
5. Prove the following properties:

   (i) z is real if and only if $z = \bar{z}$ (ii) Re (z) = $\frac{z+\bar{z}}{2}$ and Im(z) = $\frac{z-\bar{z}}{2i}$
6. Find the least value of the positive integer n for which $\left(\sqrt{3}+i\right)^{i}$

   (i) real (ii) purely imaginary
7. Show that (i) $\left(2+i\sqrt{3}\right)^{10} - \left(2-i\sqrt{3}\right)^{10}$ is purely imaginary (ii) $\left(\frac{19-7i}{9+i}\right)^{12} + \left(\frac{20-5i}{7-6i}\right)^{12}$ is real.
8. If z_1, z_2, and z_3 are complex numbers such that $|z_1| = |z_2| = |z_3| = |z_1 + z_2 + z_3| = 1$,

   find the value of $\left|\frac{1}{z1} + \frac{1}{z2} + \frac{1}{z3}\right|$.
9. Let Z_1, Z_2 and Z_3 be complex numbers such that $|Z_1| = |z_2| = |Z_3| = r > 0$ and $z_1 + z_2 + z_3 \neq 0$.

   Prove that $\left|\frac{z1z2 + z2z3 + z3z1}{z1+z2+z3}\right| = r$.
10. Find the square roots of (i) $4 + 3i$ (ii) $-6 + 8i$ (iii) $-5 - 12i$.
11. If z_1, z_2, and z_3 are three complex numbers such that $|z_1| = 1$, $|z_2| = 2$, $|z_3| = 3$ and $|z_1 + z_2 + z_3| = 1$

    Show that $|9 z_1z_2 + 4 z_1z_3 + z_2z_3| = 6$.
12. If the area of the triangle formed by the vertices z, iz, and $z + iz$ + is 50 square units, find the value of |z|.
13. Given the complex number z = 3 + 2i, represent the complex numbers z, iz, *and* $z + iz$ *in* one Argand diagram. Show that these complex numbers form the vertices of an isosceles right triangle.
14. Obtain the Cartesian form of the locus of z in each of the following cases.

    (i) |z| = |z –i| (ii) |2z – 3 –i| = 3
15. If z = x + iy is a complex number such that Im $\left(\frac{2z+1}{iz+1}\right)$ = 0, show that the locus of z is $2x^2 + 2y^2 + x - 2y = 0$.
16. Obtain the Cartesian form of the locus of $z = x + iy$ in each of the following cases.

    $[Re(iz)]^2 = 3$ (ii) Im $[(1-i)z+1] = 0$ (iii) |z + i| = |z – i| (iv) $\bar{z} = z^{-1}$
17. Show that the following equations represent a circle, and, find its centre and radius.

    |z – 2 – i| = 3 (ii) |2z + 2 – 4i| = 2 (iii) |3z – 6 + 12i| = 8.
18. Obtain the Cartesian equation for the locus of z = x + iy in each of the following cases:

    |z – 4| = 16 (ii) $|z-4|^2 - |z-1|^2 = 16$
19. Find the modulus and principal argument of the following complex numbers.

    (i) $\sqrt{3}$ + i (ii) $-\sqrt{3}$ + i (iii) $-\sqrt{3}$ – i (iv) $\sqrt{3}$ – i
20. Represent the complex number (i) – 1 – i (ii) 1 + i$\sqrt{3}$ in polar form.
21. Find the quotient $\frac{2\left(\cos\frac{9\pi}{4} + i\sin\frac{9\pi}{4}\right)}{4\left(\cos\left(\frac{-3\pi}{2}\right) + i\sin\left(\frac{-3\pi}{2}\right)\right)}$ in rectangular form.
22. If z = x + iy and $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{2}$, show that $x^2 + y^2 = 1$.

23. Find the rectangular form of the complex numbers

(i) $\left(cos\frac{\pi}{6}+i\ sin\frac{\pi}{6}\right)\left(cos\frac{\pi}{12}+i\ sin\frac{\pi}{12}\right)$ (ii) $\frac{cos\frac{\pi}{6}-i\ sin\frac{\pi}{6}}{2\left(cos\frac{\pi}{3}+i\ sin\frac{\pi}{3}\right)}$

24. If cos α + cos β + cos γ = sin α + sin β + sin γ = 0, show that

(i) cos 3α + cos 3β + cos 3γ = 3 cos(α+β+γ) and

(ii) sin 3α + sin 3β + sin 3γ = 3 sin (α+β+γ).

25. If z = x + iy and arg $\left(\frac{z-i}{z+2}\right)=\frac{\pi}{4}$, show that $x^2 + y^2 + 3x - 3y + 2 = 0$.

26. Simplify $\left(sin\frac{\pi}{6}+i\ cos\frac{\pi}{6}\right)^{18}$.

27. Simplify (i) $(1+ i)^{18}$ (ii) $(-\sqrt{3} + 3i)^{31}$

28. Solve the equation $z^3 + 8i = 0$, where $\mathbb{Z} \in \mathbb{C}$.

29. Find all cube roots of $\sqrt{3} + i$.

30. If $\omega \neq 1$ 1is a cube root of unity, show that $\frac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2}+\frac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2}$ = -1.

31. Show that $\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^5+\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^5=-\sqrt{3}$.

32. Find the value of $\left(\frac{1+sin\frac{\pi}{10}+i\ cos\frac{\pi}{10}}{1+sin\frac{\pi}{10}-i\ cos\frac{\pi}{10}}\right)^{10}$.

33. If $2\cos\alpha = x+\frac{1}{x}$ and $2\cos\beta = y+\frac{1}{y}$, show that.

(i) $\frac{x}{y}+\frac{y}{x} = 2\cos(\alpha-\beta)$

(ii) $xy-\frac{1}{xy} = 2i\sin(\alpha+\beta)$

(iii) $\frac{x^m}{y^N}-\frac{y^N}{x^m} = 2i\sin(m\alpha-n\beta)$

(iv) $x^m y^N+\frac{1}{x^m y^N} = 2\cos(m\alpha+n\beta)$.

34. Solve the equation $z^3 + 27 = 0$.

35. If $\omega \neq 1$ is a cube root of unity, show that the roots of the equation $(z-1)^3 + 8 = 0$ are $-1, 1-2\omega, 1-2\omega^2$.

36. Find the value of $\sum_{k-1}^{8}\left(cos\frac{2k\pi}{9}+i\ sin\frac{2k\pi}{9}\right)$.

37. If $\omega \neq 1$ is a cube root of unity, show that

(i) $(1-\omega+\omega^2)^6+(1+\omega-\omega^2)^6 = 128$.

(ii) $(+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)\ldots(1+\omega^{211}) = 1$.

38. If $z = 2-2i$, find the rotation of $z = by\ \theta$ radians in the counter clockwise direction about the origin when

(i) $\theta=\frac{\pi}{3}$ (ii) $\theta=\frac{2\pi}{3}$ (iii) $\theta=\frac{3\pi}{2}$.

39. Prove that the values of $\sqrt[4]{-1}$ are $\pm\frac{1}{\sqrt{2}}(1\pm i)$.

<u>Theorem with Proof</u>

40. **Property Triangle inequality**

For any two complex numbers z_1 and z_2, we have $|z_1 + z_2| \leq |z_1| + |z_2|$.

Proof

Using $|z_1 + z_2|^2 = (z_1 + z_2)\overline{(z1 + z2)}$ $(\because |z|^2 = z\bar{z})$

$= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2)$ $(\because \overline{(z1 + z2)} = \bar{z}_1 + \bar{z}_2)$

$= z_1\bar{z}_1 + (z_1\bar{z}_2 + \bar{z}_1 z_2) + z_2\bar{z}_2$

$= z_1\bar{z}_1 + (z_1\bar{z}_2 + \overline{z1 + z2}) + z_2\bar{z}_2 (\because \bar{\bar{z}} = z)$

$= |z_1^2 + \text{Re}(z_1\bar{z}_2) + |z_2|^2$ $(\because 2\,\text{Re}(z) = z + \bar{z})$

$\leq |z_1|^2 + 2\,|z_1\bar{z}_2| + |z_2|^2$ $(\because \text{Re}(z) \leq |z|)$

$= |z_1|^2 + 2|z_1||z_2| + |z_2|^2$ $(\because |z_1 z_2| = |z_1||z_2|$ and $|z| = |\bar{z}|$

$\Longrightarrow |z_1 z_2|^2 \leq (|z_1| + |z_2|)^2$

$\Longrightarrow |z_1 z_2| \leq |z_1| + |z_2|$.

41. Find the cube roots of unity.

Solution

We have to find $1^{\frac{1}{3}}$. Let z = $1^{\frac{1}{3}}$ then $z^3 = 1$.

In polar form, the equation $z^3 = 1$ can be written as

$z^3 = \cos(0 + 2k\pi) + i\sin(0 + 2k\pi) = e^{i2k\pi}$, k= 0, 1, 2, ….

Therefore, $z = cos\left(\frac{2k\pi}{3}\right) + i\ sin\left(\frac{2k\pi}{3}\right) = e^{i\frac{2k\pi}{3}}$, k = 0, 1, 2.

Taking k = 0, 1, 2, we get,

K = 0, $z = \cos 0 + i\sin 0 = 1$.

K = 1, $z = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) + i\sin\left(\pi - \frac{\pi}{3}\right)$

$= -\cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

K = 2, $z = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = \cos\left(\pi + \frac{\pi}{3}\right) + i\sin\left(\pi + \frac{\pi}{3}\right)$

$= -\cos\frac{\pi}{3} - i\sin\frac{\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

Therefore, the cube roots of unity are

$1, \frac{-i+i\sqrt{3}}{2}, \frac{-i-i\sqrt{3}}{2} \Longrightarrow 1, \omega$, and ω^2, where $\omega = e^{i\frac{2k\pi}{3}} = \frac{-i+i\sqrt{3}}{2}$.

42. Find the fourth roots of unity.

Solution

We have to find $1^{\frac{1}{4}}$. Let z = $1^{\frac{1}{4}}$ then $z^4 = 1$.

In polar form, the equation $z^4 = 1$ can be written as

$z^4 = \cos(0 + 2k\pi) + i\sin(0 + 2k\pi) = e^{i2k\pi}$, k= 0, 1, 2, ….

Therefore, $z = cos\left(\frac{2k\pi}{4}\right) + i\ sin\left(\frac{2k\pi}{4}\right) = e^{i\frac{2k\pi}{4}}$, k = 0, 1, 2, 3.

Taking k = 0, 1, 2, 3,we get,

K = 0, $z = \cos 0 + i\sin 0 = 1$.

K = 1, $z = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = i$.

K = 2, $z = \cos\pi + i\sin\pi = -1$.

K = 3, $z = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = -\cos\frac{\pi}{2} - i\sin\frac{\pi}{2} = -i$.

Fourth roots of unity are 1, i, -1, -i $\Longrightarrow$ 1, ω, and ω^2, where $\omega = e^{i\frac{2\pi}{4}} = i$.

43. Suppose z_1, z_2 and z_3, are the vertices of an equilateral triangle inscribed in the circle |z| = 2. If $z_1 = 1 + i\sqrt{3}$ then find $z2$ and $z3$.

Solution

|z| = 2 represents the circle with centre (0,0) and radius 2.

Let A, B, and C be the vertices of the given triangle. Since the vertices z_1, z_2, and z_3 form an equilateral triangle inscribed in the circle |z| = 2, the sides of this triangle AB, BC, and CA subtend $\frac{2\pi}{3}$ radians (120 degree) at the origin (circumcenter of the triangle).

(The complex number $ze^{i\theta}$ is a rotation of z by θ radians in the counter clockwise direction about the origin.)

Therefore, we can obtain z_2 and z_3 by the rotation of z_1 by $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ respectively.

Given that $\overline{OA} = z_1 = 1 + i\sqrt{3}$;

$$\overline{OB} = z_1 e^{i\frac{2\pi}{3}} = \left(1+i\sqrt{3}\right) e^{i\frac{2\pi}{3}}$$

$$= \left(1+i\sqrt{3}\right)\left(cos\frac{2\pi}{3} + isin\frac{2\pi}{3}\right)$$

$$= \left(1+i\sqrt{3}\right)\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = -2;$$

$$\overline{OC} = z_1 e^{i\frac{4\pi}{3}} = z_2 e^{i\frac{2\pi}{3}} = -2\ e^{i\frac{2\pi}{3}}$$

$$= -2\left(cos\frac{2\pi}{3} + isin\frac{2\pi}{3}\right)$$

$$= -2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 1 - i\sqrt{3}.$$

Therefore, $z_2 = -2$, and $z_1 = 1 - i\sqrt{3}$.

44. **de Moivre's Theorem**

Given any complex number cos θ + i sin θ and any integer *n*,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Corollary

(1) $(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$

(2) $(\cos\theta + i\sin\theta)^{-n} = \cos n\theta - i\sin n\theta$

(3) $(\cos\theta - i\sin\theta)^{-n} = \cos n\theta + i\sin n\theta$

(4) $\sin\theta + i\cos\theta = (\cos\theta - i\sin\theta)$

Now let us apply de Moivre’s theorem to simplify complex numbers and to find solution of equations.